EMORY MATHEMATICS DIRECTED READING PROGRAM

Program description

The Emory Math Directed Reading Program (DRP) is a graduate student-run program aiming to pair undergraduate students with graduate student mentors to read and learn material that is not typically offered in a traditional course setting. Undergraduate students are expected to work mostly independently to read the text and attempt exercises, then meet regularly with their graduate student mentor to discuss the material.

Past and Sample Topics and Descriptions

Students specify a topic (or topics) of interest when applying to the program, in order to be matched with an appropriate graduate mentor. Some past and sample topics are listed below for inspiration. See the next page for more detailed sample descriptions of the topics below.

- Algebraic number theory,
- Analysis on manifolds,
- Category theory,
- Computational number theory,
- Concrete algebra,
- Convex geometry,
- Fourier analysis,
- p-adic numbers,
- PDE Constrained Optimization,
- Percolation theory,
- Sabermetrics,
- Set theory.

If you have any questions or concerns, or suggestions for future DRP topics, please reach out to Chris Keyes at christopher.keyes@emory.edu.

Course name: Algebraic number theory
Text: *TIFR pamphlets on algebraic number theory* and *Algebraic Theory of Numbers*, by Pierre Samuel.
Prerequisites: Abstract algebra I/II (Math 421,422), Abstract vector spaces (Math 321), and some commutative algebra.

Description: The idea of the course is to amalgamate one's interest in algebra with number theory. We will read through and work out the details of the TIFR pamphlets and move on to Samuel's book from there. The goal of this short course would be to build the basics necessary to concretely understand the meaning and applications of the "Lagrange's Four Squares Theorem," which states that any natural number can be represented as the sum of four integer squares. Depending upon time and interest, we'll try to go deeper into understanding the quadratic class number formula. The pacing of the course is flexible since my goal is to make learning this topic fun!

Course name: Analysis on manifolds
Text: *Analysis on Manifolds*, by Marcello Seri.
Prerequisites: Linear algebra (Math 221), Real analysis I (Math 411). Real analysis II (Math 412) might be helpful but is not necessary.

Description: Manifolds arise naturally in many areas of mathematics, mostly in geometry and mathematical physics. They help us to generalize results from real analysis to more abstract mathematical objects. We can equip manifolds with different structures, and depending on the structure, calculus, Riemannian geometry, classical mechanics, Hamiltonian mechanics, general relativity and many different subjects can be studied.

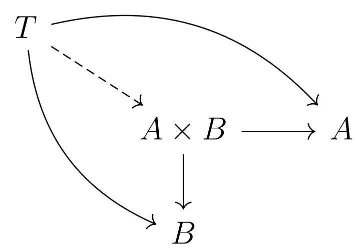
We will be interested in the structure that allows us to do calculus and analysis on these manifolds. During this course we will generalize the concepts of differentiation to smooth manifolds, study vector fields on manifolds and their relation to dynamical systems and flows. We will also study integration along curves, and depending on time and student interest, Lie groups and Lie algebras or tensor fields.

Since the theory of manifolds is a very broad subject, there are a lot of books which we could also use during this course, for example Vector Calculus by Klaus Jänich and Manifolds, Tensor Analysis, and Applications by Ralph Abraham, Jerrold E. Marsden, Tudor Ratiu.

Course name: Category theory
Text: Chapter 1 of *Foundations of Algebraic Geometry*, by Ravi Vakil, and *Categories for the Working Mathematician*, by Saunders MacLane.
Prerequisites: Abstract algebra I/II (Math 421,422).

Description: How many different kinds of "products" have you seen? In linear algebra, we take the *Cartesian product* of vector spaces, e.g. $\mathbb{R}^2 \times \mathbb{R}^3$, but we can also take the *direct product* of groups, say $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. You may also have run into products of topological spaces, graphs, or product measures. What do these constructions have in common, and why are they deserving of their designation as a *product*?

Category theory suggests that instead of studying mathetmatical objects like vector spaces, groups, and topological spaces, we study the *maps between them.* For our product example above, the categorical perspective offers the following universal definition: given objects A and B, the product $A \times B$ is the (essentially) unique object denoted $A \times B$ such that *any other* object T with maps to A and B has a map to $A \times B$.



In this course we will explore ideas like this, learning about categories, functors, universal constructions, and more. While this abstract subject is rich and fascinating in its own right, we will use plenty of examples, focusing on how a categorical perspective can help to organize and shape our view of the mathematical world.

Course name: Computational number theory
Text: *Computational Number Theory*, by Abhijit Das.
Prerequisites: Foundations of mathematics (Math 250 or equivalent). Some familiarity with programming would be helpful but not required.

Description: The field of number theory usually deals with the study of integer and rational numbers. After the recent invention of public-key cryptography, number theory became an important area of study for computer scientists. In particular, the implementation of public-key encryption algorithms, such as RSA, are built from the foundations of computational number theory. In this course we will study a range of basic number-theoretic algorithms via the computer algebra system GP/PARI. Topics include implementation of algorithms such as the Euclidean GCD algorithm, modular exponentiation, Chinese remainder theorem, Hensel lifting, quadratic residues and non-Residues, Legendre symbol, Jacobi symbol and more.

Course name: Concrete algebra
Text: *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, by David Cox, John Little, and Donal O'Shea.
Prerequisites: Linear algebra (Math 221 or equivalent).

Description: Linear algebra has taken a central role in modern mathematics due in large part to the algebraic, geometric, and computational structures inherent in linear systems of equations. When we extend our focus from linear to polynomial systems of equations, the relevant algebraic, geometric, and computational tools become more involved. In this course, we will take a hands-on approach to develop the algebra of systems of polynomial equations as well as the geometry of their solution sets. Additionally, we will see how Gaussian Elimination from linear algebra leads to a notion of *Gröbner bases* and computational methods in algebraic geometry. Throughout, we will emphasize examples which showcase how algebra, geometry, and computation are intertwined. Material can be adjusted according to student background and interest.

Course name: Convex geometry
Text: *A Course in Convexity*, by Alexander Barvinok.
Prerequisites: Linear algebra (Math 221 or equivalent), Real analysis I (Math 411) or familiarity with metric spaces.

Description: A subset C of a real vector space is convex if for any two points $x, y \in C$, the line segment connecting x and y is contained in C. For example, a pancake is convex while a donut is nonconvex. Convex sets are highly structured and appear across many areas of pure and applied math, including analysis, algebraic geometry, and optimization. The goals of this reading course will be to understand basic properties and structure of convex sets, duality theory in convex geometry, and applications to other areas of mathematics. Due to the prevalence of convexity in mathematics, applications can be highly tailored to student experience and interest.

Course name: Fourier analysis
Text: *Fourier Analysis: An Introduction*, by Stein and Shakarchi.
Prerequisites: Integral and multivariable calculus (Math 112, 211). Real analysis I (Math 411) would be helpful but not required.

Description: How does a string vibrate when you pluck it? How does heat diffuse when you turn on the stove? What is the largest area a rope of fixed length can enclose? How badly can a continuous function fail to be differentiable? Can one see an internal organ from external measurements? The study of Fourier transform and Fourier series manifests itself in various fields of mathematics such as real analysis, partial differential equation, number theory, and inverse problems.

In this reading project, we will look into basic elements of Fourier analysis and its wide applications in solving the wave equation and heat equation, obtaining the isoperimetric inequality, inverting the Radon transform, and producing an elegant proof of the Dirichlet's theorem on arithmetic progressions. Except the prerequisites, all basic materials will be developed from the scratch with various applications given depending on participants' interests.

Course name: PDE Constrained Optimization
Text: *Perspectives in Flow Control and Optimization*, by Max Gunzburger.
Prerequisites: Linear Algebra (Math 221). Differential Equations (Math 211).

Description: Many physical systems can be described using partial differential equations. To perform optimization in these systems, we need to ensure that the PDE constraints set by the physical system are obeyed. The two variables of interest in these types of problems are state variables and control variables. In fluid mechanics, the state variable might be the velocity of the fluid at a particular point, and the control variable might be the initial velocity at the inflow. The goal in this case would be to determine what inflow velocity would result in the desired velocity at the point of interest.

Transitioning from the finite to infinite dimensional setting introduces additional challenges in setting up and solving the optimization problem, but many of the tools from nonlinear optimization and numerical PDEs can still be used.

Course name: p-adic numbers
Text: *p-adic numbers: An introduction*, by F. Gouvea.
Prerequisites: Abstract algebra I/II (Math 421/422), Real analysis I (Math 411). Number theory (Math 328) would help, but isn't required.

Description: In this course, we will study p-adic numbers, which play a central role in modern number theory, laying the foundation for other topics such as class field theory and arithmetic geometry. The p-adic numbers arise from solving integer congruences, and come with interesting algebraic and analytic structures which make them useful to so many mathematicians. We will spend some time constructing and exploring the p-adics, building to key ideas such as Hensel's lemma and local-global principles. Time permitting, we may investigate further topics such as p-adic analysis and Newton polygons.

The textbook, Gouvea's *p-adic numbers: An introduction*, is a fun and conversational text with exercises sprinkled throughout. It should be an enjoyable read for students with a wide range of experience. The key prerequisite topics are groups, rings, fields, and congruences from algebra, and the basic ideas of metric topology and convergence, which should be covered in an introductory real analysis course. Experience with number theory may be helpful for motivation.

Course name: Percolation theory
Text: *Percolation*, by Béla Bollobás and Oliver Riordan.
Prerequisites: Some familiarity with probability concepts, including expected values, exposure to graph theory, and familiarity with concepts of supremum and infimum

Description: : If you submerge a porous stone in water, will its center get wet? You can model this phenomenon by thinking of the interior of the stone as a network of passages that are each either open or closed. If there is a path from the outside of the stone all the way to its center, consisting of just open passages, then the water will be able to seep in and reach the center. More generally, we can consider a grid where each pair of adjacent grid points is connected or not with some fixed probability, μ. If we have a low connection probability μ, we expect to get lots of small connected components while if we have a large

μ, we expect there to be a very large (infinite if the original grid is) component. It turns out that there will be a distinct threshold value for μ at which the behavior changes and in some cases, we can even calculate this value exactly. Percolation theory has applications to electrical networks, magnetism, and even epidemics!

Course name: Sabermetrics
Text: *Sabermetrics 101: Introduction to Baseball Analytics*, by Andy Andres.
Prerequisites: Foundations of math (Math 250).

Description: Sabermetrics is the study of baseball analytics, with a rich history dating back all the way to the origin of baseball itself. The strategies made famous in the early 21st century by the book — now also a movie — *Moneyball* have been widely adopted by MLB teams, fundamentally changing the way the game is played. Today, with advanced tracking systems (Statcast), rule changes, and (soon) robot umpires, the game is changing at a faster pace than ever before.

In this course, we will explore the foundations of modern sabermetrics to better understand the game of baseball. We will see how *run expectancy* allows us to quantify the impact of plays and players. Using this, we will be able to peek at the underpinnings of now-ubiquitous stats like *wins above replacement* (WAR) and *fielding independent pitching* (FIP), and discover why they are useful. Along the way, we will encounter ideas from probability and statistics and practice accessing the troves of publicly available baseball data using programming languages SQL and Python.

Course name: Set theory
Text: *Set Theory and Logic*, by Robert R. Stoll.
Prerequisites: Foundations of math (Math 250) or similar exposure to proofs.

Description: The theory of sets is home to some of the most philosophically challenging and counter-intuitive results in all of mathematics. The goal of this course is to explore set theory at a level deeper than is covered in Foundations of Mathematics, and our initial plan is to work through as much of Chapters 2, 3, 5, and 7 of Stoll's *Set Theory and Logic* as possible. Depending on interest and time, however, we might branch out into other readings and topics.